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Equivalence of the Regge and Einstein Equations Using Cartan's Moment of Rotation

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An explicit derivation of the Einstein tensor via Cartan's moment of rotation on an infinitesimal lattice is presented. With the standard form of the Einstein equations assumed, the equivalence of the Regge equations with matter to the Einstein equations is demonstrated in detail using a spherically symmetric example with proper time slicing. Such an example has been numerically evolved to within $r = 2M$ using null struts. These results make Regge calculus more readily applicable and provide a justification for its use.

1. THE TRANSITION FROM CONTINUOUS TO DISCRETE GEOMETRODYNAMICS

One of the various approaches to numerical general relativity which has intuitive geometric appeal is that of Regge calculus. In Regge calculus, the spacetime is discretized into blocks each of which is flat, and the variables defining the geometry are the edge lengths of the blocks. For practical applications, a slicing condition is chosen which defines a direction of time. This slicing condition amounts to a foliation of the spacetime into spacelike hypersurfaces.

The Regge equations are generally derived from an action principle. However, the usefulness of the Cartan moment of rotation for the unification of the laws of physics has recently been demonstrated (Kheyfets and Miller, 1991), and this concept has also been applied to the interpretation of the Ashtekar variables in gravitation (Kheyfets and Miller, 1992). Thus, a derivation which explicitly shows the transition from continuum geometrodynamics to the Regge equations by means of the Cartan moment of

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rotation could provide a basis for a clearer interpretation of the results of numerical procedures in general relativity.

In this paper, the Cartan moment of rotation will be shown explicitly to contain the same information as the Einstein tensor. Following this, the transition from the tensorial Einstein equations to the Regge equations with matter will be performed via a spherically symmetric example. Finally, evidence for the usefulness of the Regge equations will be provided from a recent numerical calculation.

2. DERIVATION OF THE EINSTEIN TENSOR FROM THE CARTAN MOMENT OF ROTATION

The geometric object in spacetime which produces the change in a vector parallel-transported around a planar circuit is the $({}^{1}{}_{1})$-valued curvature two-form:

$$R = 1/2\, \mathbf{e}_a \otimes \omega^b R^a{}_{bcd} \omega^c \wedge \omega^d \tag{1}$$

with the standard range, 0 to 3, of spacetime indices for all subscripts and superscripts. This operator produces a rotation of a parallel-transported vector $\mathbf{V}$ around a parallelogram $2\mathbf{A} \wedge \mathbf{B}$, since the magnitude of the vector will be preserved:

$$\Delta \mathbf{V} = -1/2\, R(\mathbf{V}, 2\mathbf{A} \wedge \mathbf{B}) \tag{2}$$

$$\Delta V^a = -V^b R^a{}_{bcd} A^c B^d \tag{3}$$

In this paper the convention for the wedge product of forms will be (Helgason, 1962; Lovelock and Rund, 1975)

$$\mathbf{a} \wedge \mathbf{b}(X_1, \ldots, X_{r+s}) = 1/(r+s)! \sum \operatorname{sign}(j_1, \ldots, j_{r+s})\, \mathbf{a}(X_{j_1}, \ldots, X_{j_r}) \mathbf{b}(X_{j_{r+1}}, \ldots, X_{j_s}) \tag{4}$$

where sign is the standard two-valued permutation operator. A corresponding definition applies to multivectors. This type of definition is naturally adapted to simplices rather than paralleleopipeds.

A more convenient form of the rotation operator for the derivation of the Einstein tensor is

$$R = 1/4\, \mathbf{e}_a \wedge \mathbf{e}_b \otimes R^{ab}{}_{cd} \omega^c \wedge \omega^d \tag{5}$$

A second object of interest, which reproduces any vector inserted into it, is

$$dP = \mathbf{e}_a \otimes \omega^a \tag{6}$$

$$\langle dP, \mathbf{A} \rangle = \mathbf{e}_a A^a = \mathbf{A} \tag{7}$$

This is sometimes called the "soldering form" or "unit tensor."

The Cartan moment-of-rotation operator, created by wedge multiplying the abstract object dP, taken as a lever arm, against the rotation operator, is directly related to the Einstein tensor (Misner *et al.*, 1973). To see this, let the Cartan moment-of-rotation trivector-valued three-form be defined as

$$\tau = dP \wedge R = 1/4\, \mathbf{e}_a \wedge \mathbf{e}_b \wedge \mathbf{e}_c \otimes R^{ab}{}_{de} \omega^c \wedge \omega^d \wedge \omega^e \tag{8}$$

It is the right and left Hodge dual of this operator which gives the Einstein tensor expressed explicitly in a tensor basis. This is now demonstrated. Applying the Hodge dual to the 3-form, we obtain

$$^*\tau = 1/4\, (\mathbf{e}_a \wedge \mathbf{e}_b \wedge \mathbf{e}_c) \otimes R^{ab}{}_{de}\, {}^*(\omega^c \wedge \omega^d \wedge \omega^e) \tag{9}$$

To compute the dual, the 3-form should be put into standard form (Straumann, 1984) to obtain the components needed which will have their indices raised:

$$\begin{aligned} \omega &= \omega^c \wedge \omega^d \wedge \omega^e \\ &= 1/3!\, \delta^{dce}{}_{ijk} \omega^i \wedge \omega^j \wedge \omega^k = 1/3!\, \omega_{ijk} \omega^i \wedge \omega^j \wedge \omega^k \end{aligned} \tag{10}$$

$$\omega_{ijk} = \delta^{dce}{}_{ijk} \tag{11}$$

$$({}^*\omega)_v = 1/3!\, \eta_{rstv} \omega^{rst} \tag{12}$$

$$\eta_{rstv} = \varepsilon_{rstv} \sqrt{g} \tag{13}$$

$$\omega^{rst} = g^{ri} g^{sj} g^{tk} \omega_{ijk} = g^{ri} g^{sj} g^{tk} \delta^{dce}{}_{ijk} \tag{14}$$

$$\begin{aligned} \eta_{rstv} \omega^{rst} &= \varepsilon_{rstv} \sqrt{g}\, g^{ri} g^{sj} g^{tk} \delta^{dce}{}_{ijk} \\ &= 3!\, \varepsilon_{rstv} \sqrt{g}\, g^{rc} g^{sd} g^{te} \end{aligned} \tag{15}$$

where g is the determinant of g_{ab} when it is looked upon as a matrix, and ε_{rstv} is the Levi-Civita density. Equation (15) is obtained by noting how the antisymmetries of the dummy indices on the Levi-Civita density and the generalized Kronecker δ symbol, also called a numerical tensor, interact. Canceling factorials, we obtain

$$({}^*\omega)_v = \varepsilon_{rstv} \sqrt{g}\, g^{rc} g^{sd} g^{te} \tag{16}$$

This object provides the components of the first dual of interest. There will also be a dual which affects the multivector. Applying now the dual to the trivector, we find

$$^*\tau^* = 1/4\, (\mathbf{e}_a \wedge \mathbf{e}_b \wedge \mathbf{e}_c)^* \otimes R^{ab}{}_{de}\, {}^*(\omega^c \wedge \omega^d \wedge \omega^e) \tag{17}$$

The standard form is again obtained,

$$\mathbf{e} = \mathbf{e}_a \wedge \mathbf{e}_b \wedge \mathbf{e}_c = 1/3!\, \delta^{mnp}{}_{abc}\, \mathbf{e}_m \wedge \mathbf{e}_n \wedge \mathbf{e}_p$$

$$= 1/3!\, e^{mnp}\mathbf{e}_m \wedge \mathbf{e}_n \wedge \mathbf{e}_p \tag{18}$$

$$e^{mnp} = \delta^{mnp}{}_{abc} \tag{19}$$

$$(\mathbf{e}^*)^q = 1/3!\, \eta^{uhwq} e_{uhw} \tag{20}$$

$$\eta^{uhwq} = -\varepsilon^{uhwq}/\sqrt{g} \tag{21}$$

$$e_{uhw} = g_{um}g_{hn}g_{wp}\, e^{mnp}$$

$$= g_{um}g_{hn}g_{wp}\delta^{mnp}{}_{abc} \tag{22}$$

Using the same kind of manipulations as above for the Levi-Civita density and the generalized δ symbol, we find

$$(\mathbf{e}^*)^q = -\varepsilon^{uhwq} g_{ua}g_{vb}g_{wc}/\sqrt{g} \tag{23}$$

The right and left Hodge dual of the moment of rotation can then be written

$$*\tau* = -1/4\, (\varepsilon^{uhwq}g_{ua}g_{hb}g_{wc})R^{ab}{}_{de}(\varepsilon_{rstv}g^{rc}g^{sd}g^{te})\mathbf{e}_q \otimes \omega^v \tag{24}$$

$$*\tau* = -1/4\, (\varepsilon^{uhwq}\varepsilon_{rstv}\delta^r{}_w g_{ua}g_{hb}g^{sd}g^{te})R^{ab}{}_{de}\, \mathbf{e}_q \otimes \omega^v \tag{25}$$

$$*\tau* = -1/4\, (\varepsilon^{ruhq}\varepsilon_{rstv})R^{st}{}_{uh}\, \mathbf{e}_q \otimes \omega^v \tag{26}$$

$$*\tau* = -1/4\, \delta^{uhq}{}_{stv}R^{st}{}_{uh}\, \mathbf{e}_q \otimes \omega^v \tag{27}$$

A straightforward evaluation of the effect of the generalized δ symbol gives

$$*\tau* = R^{uq}{}_{uv}\, \mathbf{e}_q \otimes \omega^v - 1/2\, R\mathbf{e}_q \otimes \omega^q \tag{28}$$

$$*\tau* = (R^q{}_v - 1/2\, R\, \delta^q{}_v)\mathbf{e}_q \otimes \omega^v \tag{29}$$

The Ricci tensor and scalar are now obviously displayed. The right-hand side is the Einstein tensor. Thus, the combined right and left Hodge dual of Cartan's moment of rotation (a trivector-valued three-form) is a vector-valued one-form with the same information as the Einstein tensor. This method provides an alternative approach to the standard action principle derivation as seen in Regge (1961). Although here the calculation is straightforward, there are occasions when one must carefully distinguish the two duals. The need for care and the explicit requirements for the two duals when applying the soldering form in gauge theory are documented in Kheyfets (1986), which also makes further use of Cartan's moment of rotation.

3. TRANSITION FROM THE EINSTEIN EQUATIONS TO THE REGGE EQUATIONS WITH MATTER

A form of the inhomogeneous Einstein equations which is used in Regge calculus is (Kheyfets *et al.*, 1990)

$$\sum L_j \varepsilon_j = 8\pi T_{AB} V^*{}_{AB} \tag{30}$$

where the sum on j is over the faces attached to the edge AB. Here L_j is the moment arm for the jth 2-dimensional face, called a hinge, which is attached to edge AB, ε_j is the deficit angle due to parallel transport of an arbitrary vector around hinge j, T_{AB} is the stress-energy along edge AB, and $V^*{}_{AB}$ is the volume dual to the edge AB. Referring to Fig. 1, we note that there are a total of six hinges attached to edge AB. Note that the angular displacements in the figure are orthogonal to both the radial as well as the temporal directions. This cannot be well represented in the figure. The number of hinges depends upon the type of decomposition of the spacetime. A more general tetrahedral discretization of the spacelike hypersurface at a constant value of the time parameter will have a different number of hinges per edge than the spherically symmetric discretization chosen

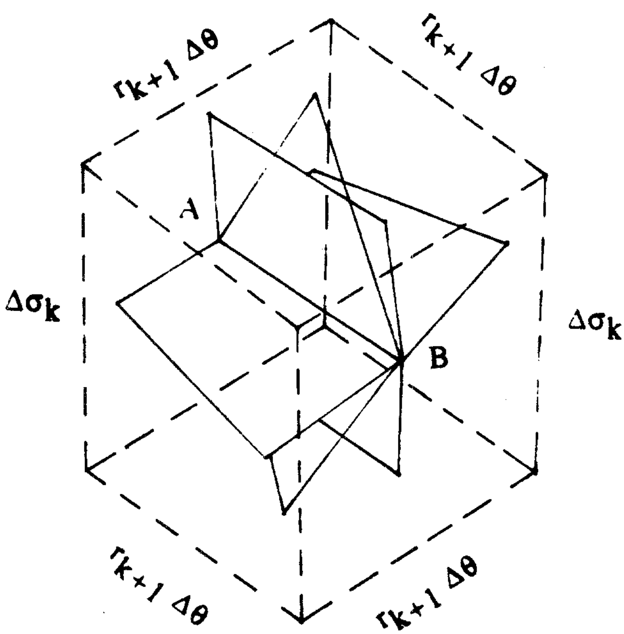

Fig. 1. Hinges and dual volume trivector for edge AB.

here. V^*_{AB} is constructed from the paths of parallel transport around the hinges. In what follows, when an explicit projection is performed, AB will be a radial edge in a spherically symmetric application of Regge calculus. It is a basic assumption of Regge calculus that the blocks of which the discretization of spacetime is made can each be given a coordinate system with an associated Minkowski metric. However, the coordinate systems cannot be made to match over a closed loop which goes through several of the blocks.

The standard Einstein equations in the style of equation (29) will now be put into the Regge equation form. After setting the right-hand side of equation (29) equal to the energy momentum tensor **T**, the dual will again be taken on the 1-form in the equation. However, on the left-hand side, the double application of the left dual on the moment of rotation will merely bring the object back to the negative of the original expression, but with the dual with respect to the multivectors remaining,

$$*\tau* = (R^a{}_b - 1/2\,\delta^a{}_b R)\mathbf{e}_a \otimes \omega^b = T^a{}_b \mathbf{e}_a \otimes \omega^b \tag{31}$$

$$**\tau* = -\tau* \tag{32}$$

$$-(\mathbf{e}_a \wedge \mathbf{e}_b \wedge \mathbf{e}_c)* \otimes R^{ab}{}_{de}(\omega^c \wedge \omega^d \wedge \omega^e) = T^a{}_b \mathbf{e}_a \otimes *\omega^b \tag{33}$$

On the right-hand side of equation (33) the dual of a one-form now appears into which the specific multivector characterizing the dual lattice volume will be inserted. The same multivector will be inserted into the left-hand side, but due to the presence of the Riemann tensor, the characteristics of the result will be different. The left-hand side will decompose into a sum of moment arms times deficit angles. This is the form of Einstein's equations which is used in Regge calculus.

The right-hand side of equation (33) will be developed to display an explicit three-form. Then, the dual volume for the case of the radial edge is inserted into it, and the resulting vector is projected onto the one-form ω^σ dual to the radial edge. The letter σ corresponds to arc length along the radial direction in a proper time slicing of the spacetime. The connecting struts between the slices are null. Afterward, the left-hand side will be developed, and similar processes will be done to it. The result will be a characteristic example of the moment arm times deficit angle Regge equations. These manipulations are now performed.

We have

$$(*\omega^b)_{stv} = \eta_{rstv}\omega^r = \varepsilon_{rstv}\sqrt{g}\, g^{rm}\omega_m \tag{34}$$

where ω_m are the *components* of the basis form ω^b,

$$\omega^b = \delta^b{}_m \omega^m \tag{35}$$

$$\omega_m = \delta^b{}_m \tag{36}$$

$$(*\omega^b)_{stv} = \varepsilon_{rstv}\sqrt{g}\, g^{rb} \tag{37}$$

$$\begin{aligned} *\omega^b &= \varepsilon_{rstv}\sqrt{g}\, g^{rb}\omega^s \otimes \omega^t \otimes \omega^v \\ &= 1/3!\; \varepsilon_{rstv}\sqrt{g}\, g^{rb}\omega^s \wedge \omega^t \wedge \omega^v \end{aligned} \tag{38}$$

Now the approximate dual volume as shown in Fig. 1 is inserted into equation (33) and the result projected onto ω^σ. Note that in this example, because of the assumptions of Regge calculus and an imaginary time coordinate, the metric is Euclidean. Here $\Delta\phi = \Delta\theta$ for convenience of calculation. $\mathbf{e}_T$ stands for the proper-time basis vector. The right-hand side of equation (33) is almost in the correct form as it stands. It only needs to be shown that $T_{AB}V^*{}_{AB}$ is the projection along the edge AB of $T^a{}_b\mathbf{e}_a \otimes *\omega^b$. Projection along AB is by the 1-form dual to $\mathbf{e}_\sigma$:

$$\begin{aligned} T^a{}_b\omega^\sigma(\mathbf{e}_a)\, *\omega^b &= T^a{}_b\,\delta^\sigma{}_a\, *\omega^b = T^\sigma{}_b\, *\omega^b \\ &= T^{\sigma r}(1/3!)\varepsilon_{rstv}\omega^s \wedge \omega^t \wedge \omega^v \end{aligned} \tag{39}$$

using the Euclidean metric. Insertion of the volume trivector dual to edge AB gives

$$\begin{aligned} &T^{\sigma r}(1/3!)\varepsilon_{rstv}\omega^s \wedge \omega^t \wedge \omega^v \\ &\quad [3!\,(i\,\Delta\sigma_k\,\mathbf{e}_T \wedge r_{k+1}\,\Delta\theta\mathbf{e}_\phi \wedge r_{k+1}\,\Delta\theta\mathbf{e}_\theta)] \\ &\quad = T^{\sigma\sigma} i\,\Delta\sigma_k\, r_{k+1}^2\,\Delta\theta^2 = T_{AB}V^*{}_{AB} \end{aligned} \tag{40}$$

By inspection of the imaginary Euclidean volume (see Figs. 1 and 2),

$$V^*{}_{AB} = i\,\Delta\sigma_k\, r_{k+1}^2\,\Delta\theta^2 \tag{41}$$

In Fig. 2, each area with a different cross-hatching denotes a volume associatd with a vertex. The diagonal lines connecting different proper times are null struts. Figure 2 suppresses all angular dimensionality and is an approximation using a Minkowski-like representation. The individual volumes actually decrease with increasing proper time. By cancellation using equations (40) and (41),

$$T_{AB} = T^{\sigma\sigma} \tag{42}$$

When the same choice of projection is made later for the left-hand side of the equation, the standard variables of the Regge equations appear naturally, so that the choice is confirmed as appropriate.

Working on the left-hand side of equation (33) containing the Riemann tensor, the operations require a bit more effort. After applying the dual operator to the trivector, inserting the dual volume trivector into the three-form, and projecting onto ω^σ, the result is

$$\begin{aligned} &\varepsilon^{ijkl} g_{ia} g_{jb} g_{kc}\,\omega^\sigma(\mathbf{e}_l) R^{ab}{}_{de}(\omega^c \wedge \omega^d \wedge \omega^e) \\ &\quad [3!\,(i\,\Delta\sigma_k\,\mathbf{e}_T \wedge r_{k+1}\,\Delta\theta\,\mathbf{e}_\phi \wedge r_{k+1}\,\Delta\theta\,\mathbf{e}_\phi)]/4 \end{aligned}$$

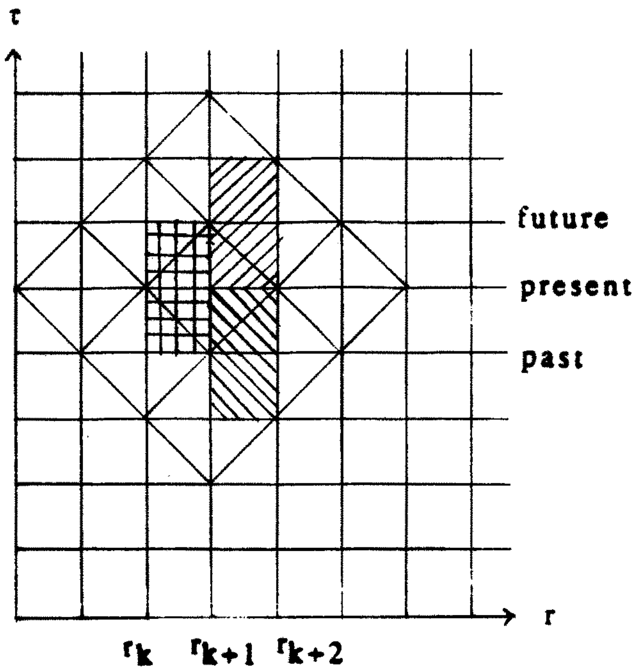

Fig. 2. Sharing of the total volume, $T-\sigma$ plane.

Examining the volume calculation separately, we find

$$\omega^c \wedge \omega^d \wedge \omega^e[3!\,(i\,\Delta\sigma_k\,\mathbf{e}_T \wedge r_{k+1}\,\Delta\theta\,\mathbf{e}_\phi \wedge r_{k+1}\,\Delta\theta\,\mathbf{e}_\theta)]$$
$$= i\,\Delta\sigma^k\,r_{k+1}^2\,\Delta\theta^2\,\delta^{cde}{}_{T\theta\phi} \tag{43}$$

It is convenient to express the numerical tensor in the following form:

$$\delta^{cde}{}_{T\theta\phi} = \delta^c{}_T\,\delta^{de}{}_{\theta\phi} + \delta^c{}_\theta\,\delta^{de}{}_{\phi T} + \delta^c{}_\phi\,\delta^{de}{}_{T\theta} \tag{44}$$

The effect of the first term from this expression will be sufficient to indicate the general result. Remembering that the metric is Euclidean, we find

$$\varepsilon^{ijkl} g_{ia} g_{jb} g_{kc}\,\delta^\sigma{}_l R^{ab}{}_{de}\,\delta^c{}_T\,\delta^{de}{}_{\phi\theta} = 2\varepsilon^{ijT\sigma} R_{ij\theta\phi} = 4R^{\theta\phi}{}_{\theta\phi} \tag{45}$$

Examining the first term generated by equation (43), we obtain the result

$$2i(\Delta\sigma_k/2)R^{\theta\phi}{}_{\theta\phi} r_{k+1}^2\,\Delta\theta^2$$

This is almost in the final form. The moment arm for the hinge in the $T-\sigma$ plane, $i\,\Delta\sigma_k/2$, appears in front of the Riemann tensor components. As there are two such hinges, one into the future and one into the past with the same value for the moment arm, the rest of the expression must involve the deficit angle. This is now shown.

As previously noted, the formula for the change in a vector parallel-transported around a small area $2(\mathbf{A} \wedge \mathbf{B})$ is

$$\Delta \mathbf{V} = -V_b R^{ab}{}_{cd} A^c B^d \mathbf{e}_a \tag{46}$$

It is known that a vector which is parallel-transported does not change its magnitude. Thus, the change of the vector has components which are normal to the original vector. A unit basis vector which is rotated a small amount in a plane will produce a component of value ε on the other basis vector of the plane. The plane is the $\theta-\phi$ plane for the radial edge and the timelike hinge of interest, and $\mathbf{e}_\theta$ is the vector to be parallel-transported. All of the projections in the space dual to the $\theta-\phi$ plane will be eliminated because they are part of the hinge. Thus, for the specific case at hand,

$$\mathbf{V} = \mathbf{e}_\theta, \qquad V^b = \delta^b{}_\theta \tag{47}$$

$$\Delta \mathbf{V} = \varepsilon \mathbf{e}_\phi = -V_b R^{\phi b}{}_{cd} A^c B^d \mathbf{e}_\phi \tag{48}$$

$$A^c = A^\theta = r_{k+1}\, \Delta\theta, \qquad B^d = B^\phi = r_{k+1}\, \Delta\theta \tag{49}$$

$$\varepsilon = -\delta_{b\theta} R^{\phi b}{}_{\theta\phi} (r_{k+1}\, \Delta\theta)^2 \tag{50}$$

$$\varepsilon = R^{\theta\phi}{}_{\theta\phi} (r_{k+1}\, \Delta\theta)^2 \tag{51}$$

Substituting this result into the first term arising from the expansion in equation (43), we find that term now reads

$$2(i\, \Delta\sigma_k/2)\varepsilon + \cdots$$

This term is of the form of moment arm times deficit angle, with the factor of 2 arising from the fact that there are two hinges of this type attached to AB, one into the future and one into the past. The other terms are evaluated in the same manner, so that one ends up with a term for each face which hinges on the edge AB, and each of these terms has the form of moment arm times deficit angle. In practice, the accurate expressions for the dual volumes should be used which are defined by the faces about which parallel transport takes place. Then one will see, for example, factors of $r^f_{k+1}\, \Delta\theta$ and $r^p{}_{k+1}\, \Delta\theta$ instead of $r_{k+1}\, \Delta\theta$ on both sides of the equation, where the f indicates a position in the future slice and p indicates a position in the past slice. This is consistent with the structure of the Cauchy problem for general relativity. These expressions can be found in Kemmell (1992). Finally, the left-hand side is set equal to the right-hand side of the original equation.

The result of the above manipulations is, in general notation,

$$\sum L_j \varepsilon_j = 8\pi T_{AB} V^*{}_{AB} \tag{52}$$

as derived for a radial strut in a spherically symmetric lattice. This particular equation is an evolution equation. Although a form of the standard Regge equations has been obtained, care must be taken using the imaginary time coordinate. In the case of the momentum constraint equations, an imaginary factor arises, and with the Hamiltonian constraint equation, a negative sign appears, as can be expected from substituting *it* for *t* in the continuum equations for the Cauchy problem in a standard text such as Adler *et al.* (1975). Arkady Kheyfets and the author both noted the need to modify the standard expression above to compensate for the use of this coordinate. However, with this proviso for this coordinate system, it can be seen that by application of the Cartan moment of rotation, the Regge equations are a discrete version of the Einstein equations.

4. RESULTS OF A NUMERICAL CALCULATION

The Regge equations with matter were used to numerically calculate Oppenheimer–Snyder collapse of pressure-free dust, using proper time slicing, null-struts, and an imaginary time coordinate. The Friedmann solution was used for analytical comparison of the interior solution, and the Novikov form of the vacuum solution was used to compare the exterior solution. These solutions were matched at the boundary of the star both analytically and numerically. Collapse to within the event horizon for a Schwarzschild black hole was modeled with 98% radial accuracy as the Schwarzschild radius was passed, with mass conserved to 90%. A full presentation of these results and a description of the numerical procedure will appear in a future paper.

5. CONCLUSION

Constructed from a product of a soldering form and the curvature 2-form, the Cartan moment of rotation is a powerful starting point for the derivation of the Einstein tensor, and it appears to have important applications in settings other than geometrodynamics. The Regge equations with matter follow directly from the Einstein equations. Application of the Regge equations to Oppenheimer–Snyder collapse shows them to be capable of dealing with massive lattices. It can be concluded that astrophysical problems using Regge lattices with matter can be solved with confidence that the fundamental approach is sound.

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